LECTURE 4.
IDEALIZED THEORY OF INTERFEROMETRIC DETECTORS — I.

Lecture by Kip S. Thorne

Assigned Reading:

A. "Gravitational Radiation" by Kip S. Thorne, in *300 Years of Gravitation*, eds. S. W. Hawking and W. Israel (Cambridge University Press, 1987), pages 414–425; ending at beginning of first full paragraph on 425. [This material uses the phrase *beam detector* for an *interferometric gravitational-wave detector*. The principal results quoted in this lecture are derived in the exercises below.]

G. The following portions of "Chapter 7. Diffraction" from the textbook manuscript *Applications of Classical Physics* by Roger Blandford and Kip Thorne: Section 7.2 (pages 7-2 to 7-7), and Section 7.5 (pages 7-20 to 7-27). [This material develops the foundations of the theory of diffraction (Green's theorem and the Helmholtz-Kirchoff formula), explores semi-quantitatively the spreading of a transversely collimated beam of light, develops the formalism of *paraxial Fourier optics* for analyzing quantitatively the propagation of collimated light beams, and uses that formalism to derive the evolution of the cross sectional shape of a Gaussian beam, of the sort used in LIGO.]

Suggested Supplementary Reading:

H. A. E. Siegman, *Lasers* (University Science Books, Mill Valley CA, 1986), chapter 17, "Physical Properties of Gaussian Beams." [This chapter develops in full detail the paraxial-Fourier-optics theory of the manipulation of Gaussian beams by a system of lenses and mirrors, and the shapes of the Gaussian modes of an optical resonator (Fabry-Perot cavity).]

A Few Suggested Problems

1. *Shot Noise.* Reread the discussion of shot noise on pages 5-20 and 5-21 of Blandford and Thorne, *Random Processes* (which was passed out last week). In that discussion let the random process $y(t)$ be the intensity $I(t) = d(\text{energy})/dt$ of a laser beam, and let $F(t)$ be the intensity carried by an individual photon, which has frequency ω.
   (a) Explain why $\tilde{F}(0)$, the Fourier transform of F at zero frequency, is the photon energy $\hbar\omega$.
   (b) Show that the spectral density of I (the "shot-noise spectrum") is
   $$G_I(f) = 2\bar{I}\hbar\omega \,, \tag{1}$$
   where $\bar{I}$ is the beam's mean intensity.
   (b) Let $N(t)$ be the number of photons that the beam carries into a photodiode between time t and time $t+\hat{\tau}$ (so $\hat{\tau}$ is the averaging time): $N(t) = \int_t^{t+\hat{\tau}} I(t')dt'$.

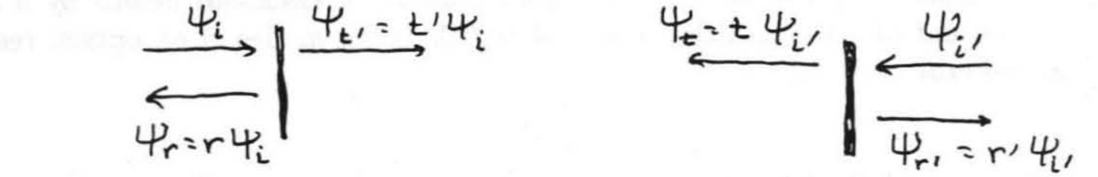
This $N(t)$ is a linear functional of $I(t')$. Use the theory of linear signal processing to derive the spectral density $G_N(f)$ of $N(t)$, and then compute the mean square fluctuations of N: $(\sigma_N)^2 = \int_0^\infty G_N(f)df$. Your result should be $\sigma_N = \sqrt{\bar{N}}$, where $\bar{N}$ is the mean number of photons that arrive in the averaging time $\hat{\tau}$. This is the standard "square-root-of-N" fluctuation in photon arrival for a laser beam.

2. *Reciprocity Relations for a Mirror and a Beam Splitter.* Modern mirrors, beam splitters, and other optical devices are generally made of glass or fused silica (quartz), with dielectric coatings on their surfaces. The coatings consist of alternating layers of materials with different dielectric constants, so the index of refraction n varies periodically. If, for example, the period of n's variations is half a wavelength of the radiation that impinges on the device, then waves reflected from successive dielectric layers build up coherently, producing a large net reflection coefficient. In this exercise we shall derive the reciprocity relations for a mirror of this type, with normally incident radiation. The generalization to radiation incident from other directions, and to other dielectric optical devices is straightforward.

The foundation for the analysis is the wave equation,

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{c^2}{n^2(\mathbf{x})}\nabla^2\right)\psi = 0$$

satisfied by any Cartesian component ψ of the electric field, and the assumption that ψ is precisely monochromatic with angular frequency ω. These imply that the spatial dependence of ψ is governed by the Helmholtz equation with spatially variable wave number $k(\mathbf{x}) = n(\mathbf{x})\omega/c$: $\nabla^2\psi + k^2\psi = 0$.



Let waves $\psi_i e^{ikz}$ impinging perpendicularly (z direction) on the mirror from the "unprimed" side produce reflected and transmitted waves $\psi_r e^{ikz}$ and $\psi_{t'} e^{ikz}$; these waves and their corresponding ψ inside the mirror are one solution ψ_1 of the Helmholtz equation. The complex amplitudes of this solution are related by reflection and transmission coefficients, $\psi_r = r\psi_i$, $\psi_{t'} = t'\psi_i$. Another solution, ψ_2, consists of incident waves from the opposite, "primed" side, $\psi_{i'}e^{-ikz}$ and reflected and transmitted waves $\psi_{r'}e^{+ikz}$, $\psi_t e^{-ikz}$, and the corresponding ψ inside the mirror; and this solution's complex amplitudes are related by $\psi_{r'} = r'\psi_{i'}$, $\psi_t = t\psi_{i'}$.

(a) Show that ψ obeys Green's theorem [Equation (7.3) of Blandford and Thorne] throughout the mirror. Apply Green's theorem, with ψ and ψ_0 chosen to be various pairs of ψ_1, ψ_2, ψ_1^*, ψ_2^* (where the star denotes complex conjugation). Thereby obtain four relationships between r, r', t, and t'.

(b) Show that these relationships can be written in the form

$$r = \sqrt{\mathcal{R}}e^{2i\beta}, \quad r' = -\sqrt{\mathcal{R}}e^{2i\beta'}, \quad t = t' = \sqrt{\mathcal{T}}e^{i(\beta+\beta')}\ ,$$

where β and β' are unconstrained phases, and $\mathcal{R}$ and $\mathcal{T}$, the power reflection and transmission coefficients are related by

$$\mathcal{R} + \mathcal{T} = 1 \,, \tag{2}$$

which is just energy conservation.

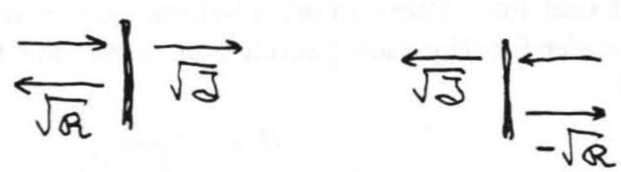
(c) Show that, if one moves the origin of coordinates as seen from the unprimed side by $\delta z = -k\beta$, and moves the origin as seen from the primed side by $\delta z = +k\beta'$, one thereby will make all the reflection and transmission coefficients real:

$$t = t' = \sqrt{\mathcal{T}}, \quad r = -r' = \sqrt{\mathcal{R}} \,. \tag{3}$$

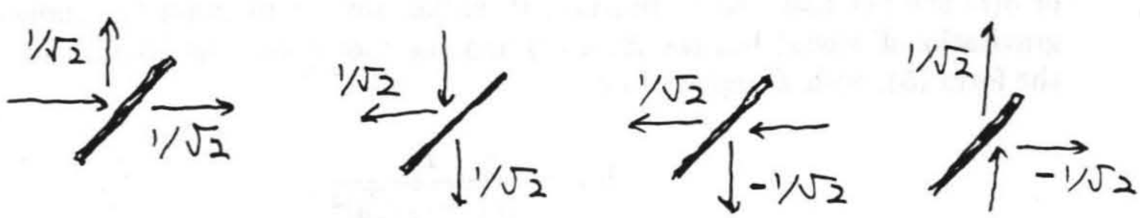
Thus, with an appropriate choice of origin on each side of the mirror, the coefficients can always be made real.

The same is true for the reflection and transmission coefficients of any other optical device made of a lossless, spatially variable dielectric. In particular, for a perfect, 50/50 beam splitter, the transmission coefficient becomes, with appropriate choice of origins, $1/\sqrt{2}$ from each and every one of the four input ports, and the reflection coefficient becomes $+1/\sqrt{2}$ from the input ports on one side of the beam splitter and $-1/\sqrt{2}$ from the input ports on the other side of the beam splitter. These results are summarized by the following figures:

mirror :



beam-splitter



2. *Transfer Function and Photon Shot Noise for a Delay-Line Interferometer* In class, Kip derived the "tranfer function" for a delay-line interferometer in the limiting regime where the waveform $h(t)$ is nearly constant during the time $2BL/c$ that the light is stored in the interferometer arms (during B round trips in an arm whose length is L). His result was

$$I_{PD}(t) = I_1(t) + 2\sqrt{\bar{I}_1 I_0} BkLh(t) \tag{4}$$

where I_0 is the mean laser input power entering the beamsplitter, $I_1(t)$ is the (slightly fluctuating because of shot noise) intensity of the light falling onto the photodiode in the absence of a gravitational-wave signal, $\bar{I}_1$ is the mean intensity onto the photodiode, B is the number of round trips in the arms of the interferometer, $k = \omega/c = 2\pi/\lambda_e$ is the light's wave number, L is the arm length, and $h(t)$ is the gravitational waveform. Kip used this and the shot-noise spectral density [Eq. (1) above] to derive the following expression for the shot-noise contribution to the interferometer's gravitational-wave noise output:

$$G_h(f) = \frac{\hbar\omega}{2I_0(BkL)^2} \,. \tag{5}$$

(a) Use the same method of analysis as Kip did in class to derive the transfer function when the gravitational wave is sinusoidal in time with angular frequency $\Omega = 2\pi f$, i.e. when $h(t) = h_o \cos(\Omega t) = h_o \mathrm{Real}(e^{-i\Omega t})$, with a frequency f high enough (gravitational wavelength short enough) that the waveform *can* vary significantly while the light is stored in the arms. Your result should be the same as Eq. (4), with B replaced by

$$B_{\rm eff} = B\frac{\sin(f/f_0)}{f/f_0}\,, \quad f_0 \equiv \frac{c}{2\pi BL} = \frac{119\mathrm{Hz}}{(B/100)(L/4\mathrm{km})} \,. \tag{6}$$

(b) Show that the shot-noise contribution to $G_h(f)$ has the form (5) with B replaced by $B_{\rm eff}$.

3. *Transfer Function and Photon Shot Noise for a Fabry-Perot Interferometer.* In class, Kip showed that for a Fabry-Perot interferometer in the regime of slow variations of $h(t)$ the transfer function and photon shot noise have the forms (4) and (5), with B replaced by

$$B_{\rm eff} = \frac{4}{(1-\mathcal{R})} \tag{7}$$

where $\mathcal{R}$ is the power reflectivity of the interferometer's corner mirrors and where it is assumed that the end mirrors are perfectly reflecting. Show that, if the variations of $h(t)$ are not assumed to be slow, then the transfer function (for monochromatic gravitational waves) has the form (4) and the shot noise contribution to $G_h(f)$ has the form (5), with B replaced by

$$B_{\rm eff} = \frac{B}{\sqrt{1+(f/f_0)^2}}\,, \tag{8}$$

where f_0 is as in Eq. (6) above.

Lecture 4
Idealized Theory of Interferometers — I.

by Kip S. Thorne, 8 April 1994

Thorne lectured at the blackboard. The following are the notes from which he lectured, cleaned up a bit to make them more understandable.

Kp Thorne
Ph 106

Lecture 4 — Idealized Theory of Interferometers

1. Overview of How an Interferometer Works & Orders of Magnitude

a. **Goal:** $h \sim 10^{-22}$; $\frac{\Delta L}{L} = h$; $L = 4\,\text{km} \sim 10^6\,\text{cm}$

$\Rightarrow \Delta L \sim 10^{-16}\,\text{cm}$; $\lambda_e \simeq 0.5\,\mu m \approx 10^{-4}\,\text{cm}$

$\Rightarrow$ measure 10^{-12} of λ_e ! seems outrageous

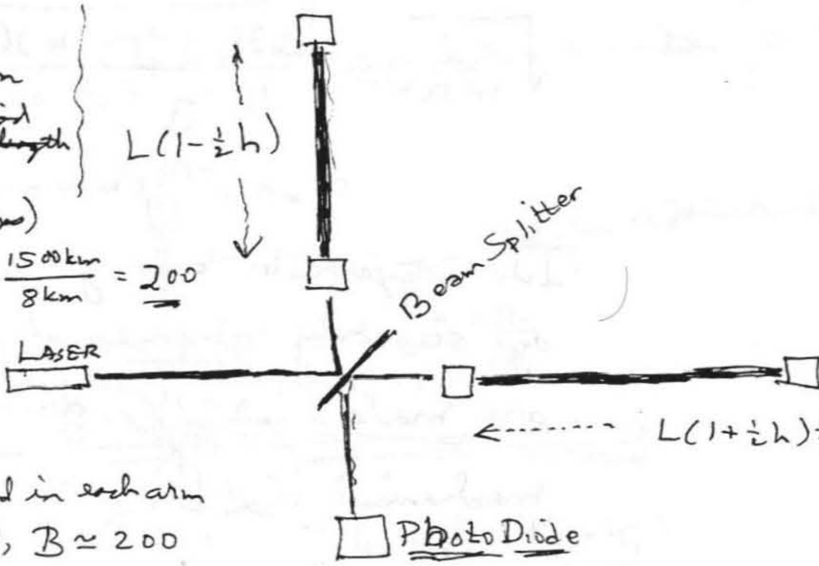
b. GW $f \sim 100\,\text{Hz}$

$\lambda \sim 3000\,\text{km}$

(light) store for $\frac{1}{2}$ period ~~wavelength~~

$\Rightarrow B =$ (# of round trips)

$= \frac{\lambda/2}{2L} = \frac{1500\,\text{km}}{8\,\text{km}} = 200$



(So light is stored in each arm for, on average, $B \simeq 200$ round trips)

c. Phase shift: $\Delta\Phi = \underbrace{\frac{2\pi}{\lambda_e}}^{k} 2B\Delta L \simeq 200 \times \frac{2\pi}{5\times10^{-5}\,\text{cm}} 2\times10^{-16}\,\text{cm}$

$\simeq 10^{-9}$

d. How accurately can this phase shift be measured?

If clever & good: $\Delta\Phi \simeq \frac{1}{\sqrt{N_\gamma}}$ $\left[\text{Proof: } \Delta E \Delta t \gtrsim \hbar \Rightarrow \underbrace{\Delta N}_{\sqrt{N_\gamma}} \underbrace{\omega \Delta t}_{\Delta\Phi} \gtrsim 1\right]$ ($\Delta E = \hbar\omega\Delta N_\gamma$)

N_γ = number of photons from laser in time $1/f = 0.01\,\text{sec}$

$\Rightarrow$ need 10^{18} photons in $0.01\,\text{sec}$ [Photon shot noise]

$\Rightarrow I = \frac{10^{18} \times \hbar\omega}{10^{-2}\,\text{sec}} \simeq 5\times10^{15-27+18+2} \simeq 5\times10^{8}\,\frac{\text{erg}}{\text{s}}$ ($\hbar\omega = 10^{-27} \times 5\times10^{15}$)

$\simeq$ 50 Watts

[Can be achieved w/ 5 Watt laser and a 10-fold recycling of used light.]

e. Won't vibrations of atoms in mirror prevent measurement of such tiny motions? No –

i. Individual atoms vibrate at $f \sim 10^{13}$ Hz, far above interferometers' gravity-wave band

ii. Only concern is lowest frequency normal modes which have thermal amplitudes

$$\sqrt{\frac{kT}{m\Omega^2}} \simeq \sqrt{\frac{(1.38\times10^{-16}\mathrm{erg/K})(300\mathrm{K})}{10^4\mathrm{g}\ (10^5/\mathrm{s})^2}} \simeq \sqrt{4\times10^{-16+2-4-10}}$$

$$\simeq 2\times10^{-14}\ \mathrm{cm}.$$

The interferometer averages over many periods and sees only changes of amplitude – which are made small by giving mirrors high mechanical Q's.

This thermal noise will be discussed in later lectures.

2. Ways to Produce Multiple Bounces in Interferometer Arms

a. Delay Line [many ~~ten~~ discrete spots on each mirror]

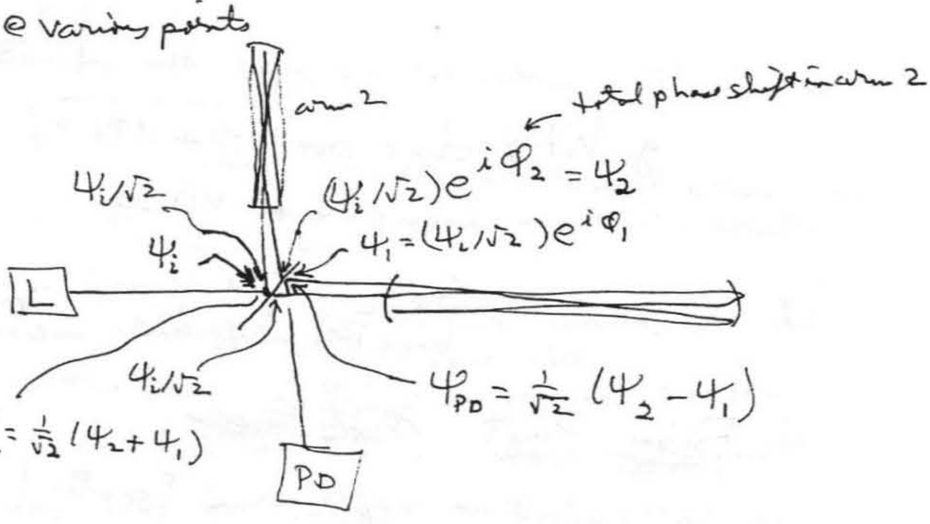
b. Fabry-Perot [one spot on each mirror]

3. Delay Line — ω h constant during storage time : $2BL << \lambda_{GW}/2$

a. Describe light by $\psi = \frac{E_x}{\sqrt{4\pi}}$ ... so $\boxed{I = |\psi|^2}$

$\psi = \psi e^{-i\omega t}$; $\psi = e^{ikx}$ if propagate in x direction ; $\boxed{k = \frac{\omega}{c} = \frac{2\pi}{\lambda_e}}$

b. Field @ various points



c. A bit of algebra!

$\psi_L = \frac{1}{2}\psi_i\left(e^{i\varphi_2} + e^{i\varphi_1}\right)$ [is field going back toward laser from interferometer]

$\Rightarrow \boxed{|\psi_L|^2 = \underbrace{|\psi_i|^2}_{I_0} \cos^2\left(\frac{\varphi_2 - \varphi_1}{2}\right)}$ intensity toward laser

$\psi_{PD} = \frac{1}{2}\psi_i\left(e^{i\varphi_2} - e^{i\varphi_1}\right) \leftarrow$ [Field going toward photodiode]

$\Rightarrow \boxed{|\psi_{PD}|^2 = \underbrace{|\psi_i|^2}_{I_0} \sin^2\left(\frac{\varphi_2 - \varphi_1}{2}\right)}$ intensity toward PD

d. Operate PD on "dark fringe" ["dark port"] $\boxed{I_0 = \text{Laser Power}}$

so $\boxed{\varphi_2 - \varphi_1 \equiv \varphi_0 << 1 \text{ before wave arrives}}$

e. Effect of wave:

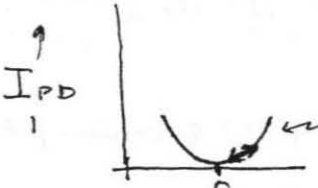
$$\boxed{\delta\varphi_2 = -\delta\varphi_1 = k\cdot \overbrace{\tfrac{h}{2}L}^{\delta L \text{ in each arm}} \cdot \overbrace{2B}^{\#\text{ of one way trips}}}$$

$\boxed{B = \#\text{ of round trips in each arm}}$

$$\delta\varphi = \tfrac{1}{2}(\delta\varphi_2 - \delta\varphi_1)$$

$$\boxed{\delta\varphi = B\cdot 2k\,\delta L = B\cdot kLh}$$

f. $\boxed{I_{PD}(t) = I_0\left(\varphi_0^2 + 2\varphi_0 BkL\,h(t)\right) \quad \text{If } 2BL \ll \lambda_{GW}/2}$ ← Intensity of light into photodiode



as h oscillates, $\delta\varphi$ oscillates, and I_{PD} oscillates up and down on side of parabola

g. WHY dark port toward PD?

– Keep power on PD low

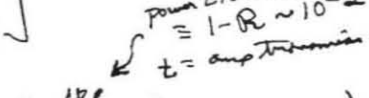
– Send light toward laser, so it can be recycled back into interferometer with new laser light

4. Fabry-Perot: ~~Single Cavity~~

a. Look at a single arm [cavity]

b. How it gets excited:

power transmission $= 1-R \sim 10^{-2}$
t = amp transmission $\sim 1/10$



i. Turn on light suddenly, on resonance so $2kL$ = multiple of 2π

ii. First pass of light down arm

$\psi = t\psi_i$ [t = amplitude transmission]

returns in phase ⇒ next pass ~~$t r \psi_i$~~ $r(t\psi_i)$

returns in phase again ⇒ next: ~~t~~ $r^2(t\psi_i)$

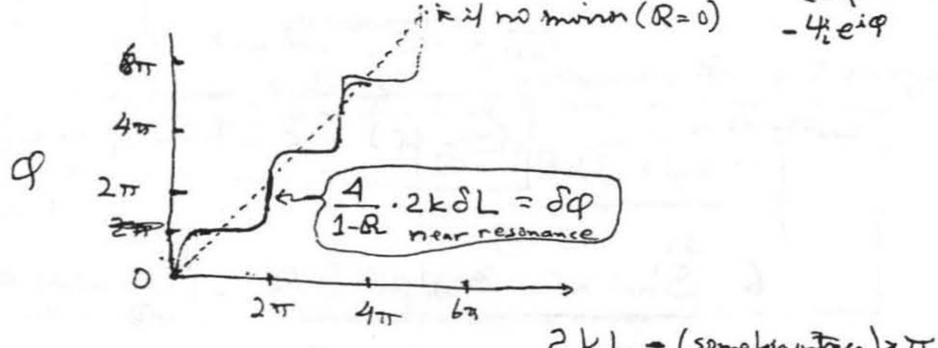
⋮

$$\psi_{inside} = t\psi_i(1+r+r^2+\dots) = \frac{t\psi_i}{1-r} \simeq \frac{\sqrt{1-r^2}\,\psi_i}{\frac{1}{2}(1-r^2)}$$

$$\psi_{inside} = \frac{2\psi_i}{\sqrt{1-R}}\ ; \quad \boxed{I_{inside} = \frac{4}{1-R} I_0}$$

c. If off resonance: cannot excite cavity with laser light

d. Phase shift as function of length of cavity (arm) $\underset{-\psi_i e^{i\varphi}}{\overset{\psi_i}{\rightleftarrows}}$(- - - - -)



... i.e. $\frac{4}{1-R}$ plays role of the "Q"

e. Compare with delay line. — Same, with

$\boxed{B = \frac{4}{1-R}}$ ... effective number of bounces in a Fabry-Perot interferometer.

5. Shot Noise [cf. Random Process Chapter, Ref. D]

a. The beam $I_{PD} = I_0\varphi_0^2$ (intensity toward photodiode) — in absence of GW's — consists of photons which arrive randomly at photodiode. Each photon carries energy $\hbar\omega$, so average rate of arrival is $R = \frac{I_{PD}}{\hbar\omega} \sim (10^{18}/\text{sec})\frac{I}{1W}$

Since duration τ_p of each photon pulse is $> 10^{-15}$ sec, $\boxed{R\tau_p \gg 1}$

b. This random arrival means I_{PD} fluctuates randomly, $I_{PD}(t)$, @ some spectral density $G_{I_{PD}}(f)$.

c. GW Frequencies of interest to us, $f \sim 0.01$ sec, are $\ll \frac{1}{\tau_p}$.
At these frequencies, the shapes of the pulses cannot influence $G_{I_{PD}}(f)$ — low f limit!! $\Rightarrow \boxed{G_I(f) \simeq G_I(0) = \text{const}}$

d. Exercise: From requirement that if $\bar{N} = \frac{I}{\hbar\omega}\hat{\tau}$ is mean # that arrive during time $\hat{\tau}$, then $\sigma_N = \sqrt{\bar{N}}$, we get

$$\boxed{G_I(f) = 2\bar{I}\hbar\omega} \quad \text{— Check unit : } \frac{I^2}{Hz}$$

6. Short noise ~~limit~~ in $h(t)$: (Translate this photon shot noise into an equivalent noise in grav. wave signal)

a. $I_{PD}(t) = I_0 [\Phi_0^2 + 2\Phi_0 BkL\,h(t)]$

b. Rewrite formula on page 4 with $I_0\Phi_0^2 = I_1(t)$; $I_0\Phi_0 = \sqrt{I_0 \bar{I}_1}$

$$\boxed{I_{PD} = I_1(t) + 2\sqrt{\bar{I}_1 I_0}\,BkL\,h(t)}$$

Then $G_{I_1}(f) = 4\bar{I}_1 I_0 (BkL)^2 G_h(f)$

$$\Rightarrow \boxed{G_h(f) = \frac{G_{I_1}(f) \leftarrow 2\bar{I}_1\hbar\omega}{4\bar{I}_1 I_0 (BkL)^2}}$$

c. $$\boxed{G_h(f) = \frac{\hbar\omega}{2I_0(BkL)^2} \quad \text{... white noise spectrum}}$$

d. $$\boxed{h_{rms} = \sqrt{f\,G_h(f)} = \underbrace{\frac{1}{2BkL}}_{\frac{\lambda_e}{2BL}} \cdot \frac{1}{\sqrt{\underbrace{(I_0/\hbar\omega)(1/2f)}_{P/2 =}}}}$$

# of photons available in 1/2 GW period

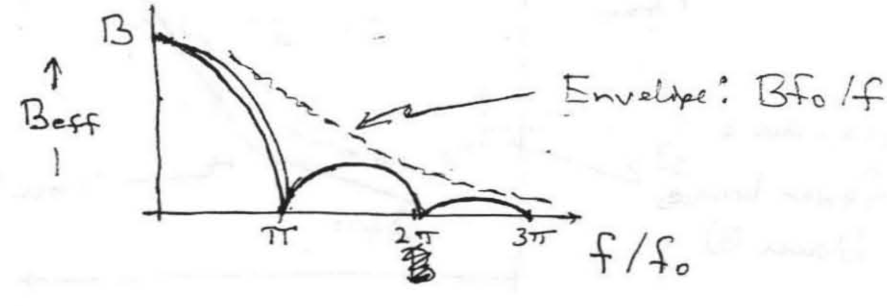
These are the shot noise limits on our interferometer — valid both for Delay Line and Fabry Perot.

7. What if $\lambda_{GW} \lesssim 2BL$?

a. Delay Line: ~~[illegible]~~

put shift onto light, then remove, then put on again...

$$\Rightarrow \boxed{B_{eff} = B \cdot \left| \frac{\sin\left(B \cdot \overbrace{(2\pi f L/c)}^{2\pi L/\lambda_{GW}}\right)}{B \cdot 2\pi f L/c} \right| = B\,\frac{\sin(f/f_0)}{f/f_0}}$$



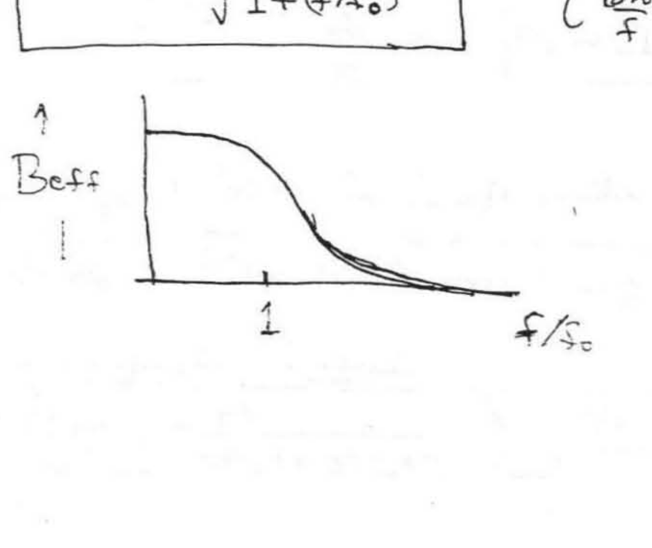
$$\boxed{f_0 = \frac{c}{2\pi B L} = \frac{11\,9\,Hz}{(B/100)(L/4km)}}$$

$$\boxed{\frac{Bf_0}{f} = \frac{c}{2\pi L f} = \frac{\lambda_{GW}}{L}}$$

b. Fabry-Perot:

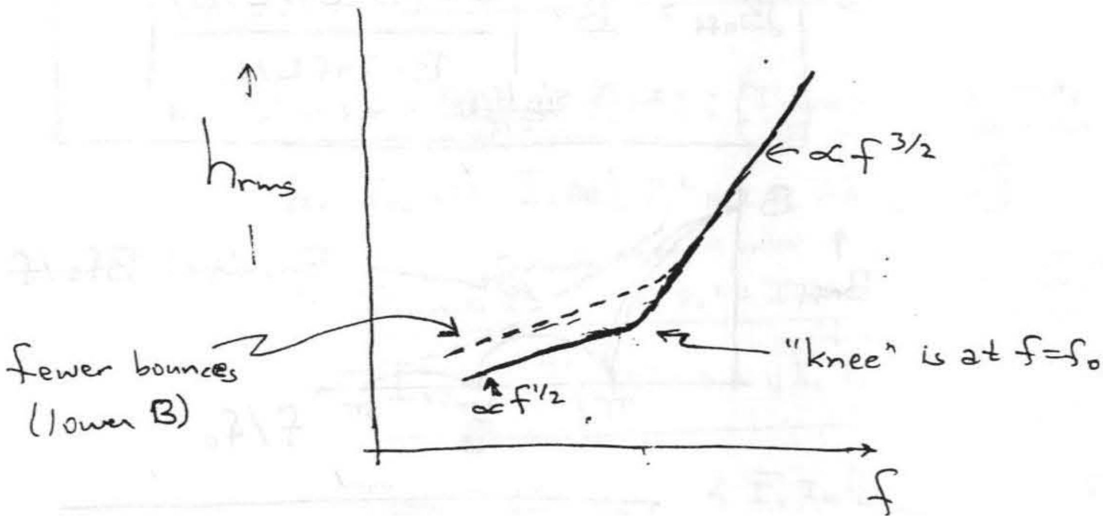
photons stored for a statistically varying length of time $\Rightarrow$ smooth die out of B:

$$\boxed{B_{eff} = \frac{B}{\sqrt{1+(f/f_0)^2}}} \simeq \begin{cases} B & f \ll f_0 \\ \frac{Bf_0}{f} = \frac{2\pi L}{cf} = \frac{L}{\lambda_{GW}} & f \gg f_0 \end{cases}$$



8. Bottom Line

$$\boxed{h_{rms} = \sqrt{f\,G_h(f)} = \frac{\lambda_e}{2BL}\,\frac{1}{\sqrt{(I_0/\hbar\omega)(1/2f)}}\cdot\sqrt{1+(f/f_0)^2}}$$

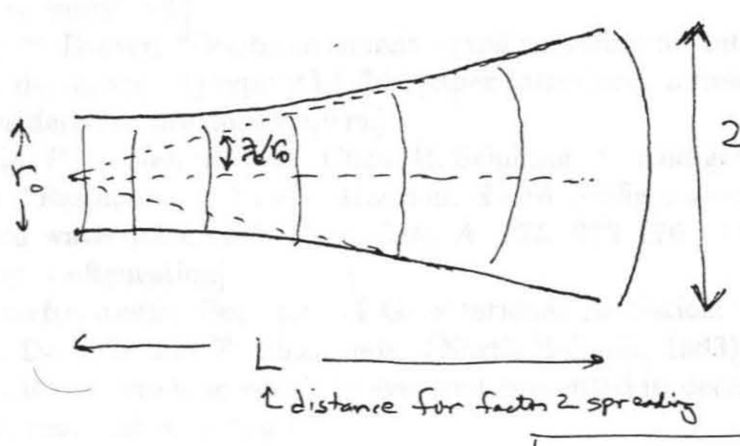


9. Gaussian Beams & Paraxial Optics [§7.5 of Ref. G]

a. Basic idea of Wave-spreading

$$\Delta k_y r_o \sim 1$$

$$\Delta k_y \sim \frac{1}{r_o} \qquad \frac{\Delta k_y}{k} \sim \frac{\bar{\lambda}_e}{r_o} = \text{opening angle}$$



$$\frac{2r_o}{L} \sim \frac{\bar{\lambda}_e}{r_o} \Rightarrow \boxed{r_o \sim \sqrt{L\bar{\lambda}_e}}$$

b. This fixes size of beam in interferometer arms:

$$r_o \sim \sqrt{L\bar{\lambda}_e} \sim \sqrt{(4\times10^5\,\text{cm})(4\times10^{-5}\,\text{cm})} \sim 4\,\text{cm}$$

c. Transverse profile is Gaussian

$$\boxed{\Psi \sim \exp\left(-\frac{r^2}{r_o^2[1+z^2/z_o^2]}\right)} \quad \boxed{z_o = r_o^2/\bar{\lambda}_e}$$

d. Beam is matched into cavity using lenses that manipulate its radius and its radius of curvature of phase fronts.

LECTURE 5.
IDEALIZED THEORY OF INTERFEROMETRIC DETECTORS—II.

Lecture by Ronald W. P. Drever

Assigned Reading:

I. R. W. P. Drever, "Fabry-Perot cavity gravity-wave detectors" by R. W. P. Drever, in *The Detection of Gravitational Waves*, edited by D. G. Blair (Cambridge University Press, 1991), pages 306–317. [This is a qualitative overview of Fabry-Perot gravitational-wave detectors, with emphasis on recycling in the later part (pages 312—317).]

J. B. J. Meers, "Recycling in laser-interferometric gravitational-wave detectors," *Phys. Rev. D*, **38**, 2317–2326. [This is the paper in which Meers introduced his idea of dual recycling and sketched out its features. You are not expected to master all the equations in this paper—which Meers just gives without derivation—but you might try deriving some of the equations as a homework exercise.]

Suggested Supplementary Reading:

K. B. J. Meers, *Physics Letters A*, "The frequency response of interferometric gravitational wave detectors," *Physics Letters A*, **142**, 465 (1989). [In this paper Meers discusses in some detail the frequency responses and sensitivities of various configurations of recycled interferometers.]

L. B. J. Meers and R. W. P. Drever, "Doubly-resonant signal recycling for interferometric gravitational-wave detectors." (preprint) [This paper introduces a new recycling configuration, not considered in previous papers.]

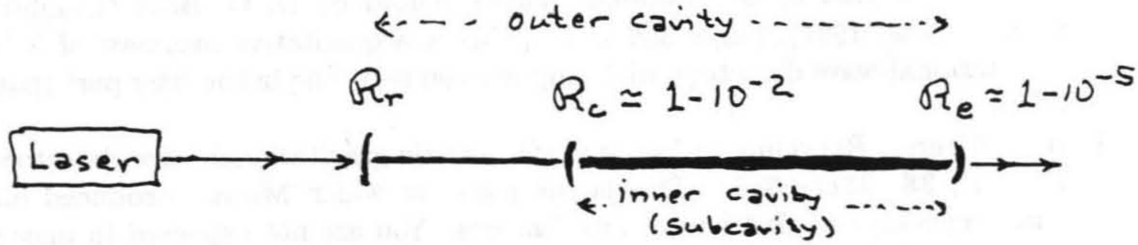
M. J. Mizuno, K. A. Strain, P. G. Nelson, J. M. Chen, R. Schilling, A. Rudiger, W. Winkler and K. Danzman, "Resonant sideband extraction: a new configuration for interferometric gravitational wave detectors," *Phys. Lett. A*, **175**, 273–276 (1993). [This is yet another recycling configuration]

N. R. W. P. Drever, "Interferometric Detectors of Gravitational Radiation," in *Gravitational Radiation*, N. Deruelle and T. Piran, eds. (North Holland, 1983); section 8 (pages 331-335). [This is the article in which Drever first presented in detail his ideas of power recycling and resonant recycling.]

A Few Suggested Problems

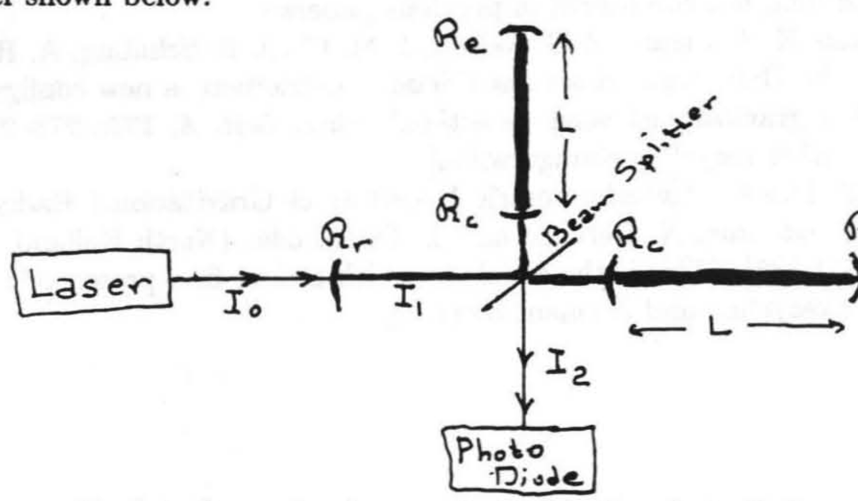
Note: Of all configurations for a recycled interferometer, the only one that is reasonably easy to analyze is power recycling. For this reason, and because this is the type of recycling planned for the first LIGO interferometers, I have chosen to focus solely on power recycling in the following exercises. — Kip.

1. *Simplified Configuration of Nested Cavities that Illustrates Power Recycling:* Consider the configuration of two nested optical cavities shown below:



All three mirrors are assumed ideal in the sense that they do not scatter or absorb any light; therefore each of them satisfies the reciprocity relations of Assignment 4, Eq. (3). Assume that the power reflectivities of the subcavity are fixed: $\mathcal{R}_e$ is the highest reflectivity the experimenter has available; $\mathcal{R}_c$ is a much lower reflectivity, carefully designed to store the light in the subcavity for a chosen length of time. What reflectivity $\mathcal{R}_r$ should the recycling mirror have in order to maximize the light intensity in the subcavity, when both cavities are operating on resonance? Use physical reasoning to guess the answer before doing the calculation.

2. *Optimization of a Power Recycled Interferometer.* Consider the power-recycled interferometer shown below.



   a. Suppose the interferometer is operated with the photodiode very near a dark fringe, so the light power I_2 is many orders of magnitude less than I_1. As in exercise 1, let $\mathcal{R}_e$ and $\mathcal{R}_c$ be fixed. How should $\mathcal{R}_r$ be chosen to maximize the power in the interferometers' two arms? Guess the answer on physical grounds before doing the calculation.

   b. Again, suppose that I_2 is many orders of magnitude less than I_1. Let a low-frequency gravitational wave (one with $2\pi f BL/c \ll 1$ where $B = 4/(1-\mathcal{R}_c)$ is the effective number of round trips in the arms) impinge on the interferometer. How should $\mathcal{R}_r$ be chosen so as to maximize the gravitational-wave signal to noise ratio in the interferometer? Guess the answer on physical grounds before doing the calculation.

c. Suppose that the mirrors in the two arms are slightly imperfect, and their imperfections cause a mismatching of the phase fronts of the light from the two arms at the beam splitter. As a result, the ratio $I_2/I_1 \equiv \alpha$ has some modest value (e.g. 0.01) instead of being arbitrarily small. In this case, how should $\mathcal{R}_r$ be chosen so as to maximize the signal to noise ratio? Guess the answer on physical grounds before doing the calculation.

3. *Scaling of Photon Shot Noise with Arm Length.* We saw in Kip's lecture that, if one has mirrors of sufficiently high reflectivity and one uses a simple (nonrecycled) interferometer, then the photon shot noise $h_{\rm rms} = \sqrt{f G_h(f)}$ is independent of the interferometer's arm length.

   Suppose, instead, that (i) the highest achievable power reflectivity is $\mathcal{R} = 1-10^{-5}$, (ii) one can do as good a job of phase-front matching at the interferometer as one wishes, so in the above drawing $I_2/I_1 = \alpha$ can be made as small as one wishes, (iii) one has a fixed laser power I_0 (say, 10 Watts) available, (iv) one operates the interferometer in a power-recycled mode, as in the above figure. *Show* that in this case the photon shot noise $h_{\rm rms}$ scales as $1/\sqrt{L}$ in the full LIGO frequency band (a result quoted on page 314 of Ref. I).

   Note: Another example of arm-length scaling is described on page 316 of Ref. I: A resonant-recycled or dual-recycled interferometer looking for periodic gravitational waves, e.g. from a pulsar, has photon shot noise $h_{\rm rms} \propto 1/L$.

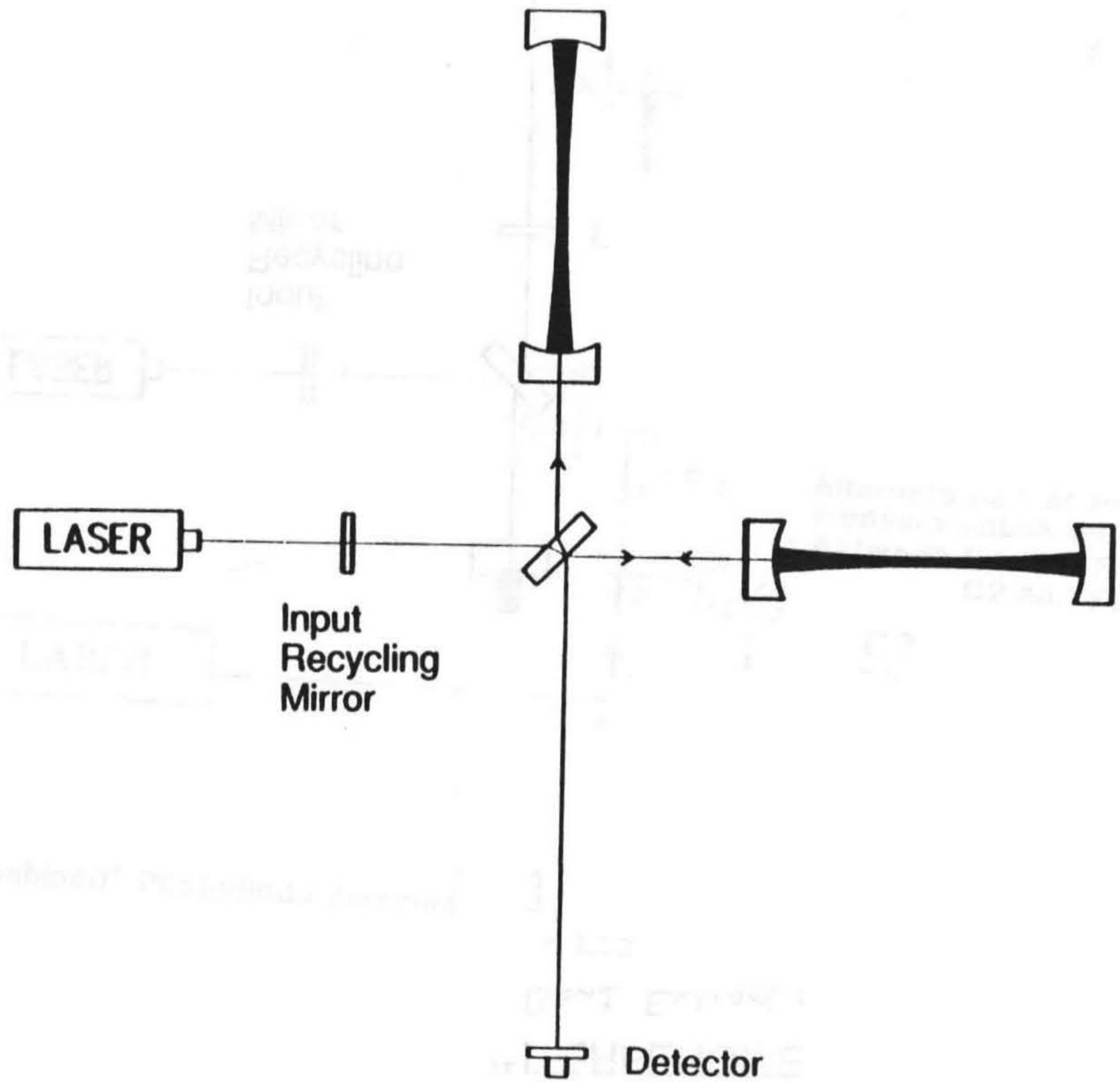
Lecture 5
Idealized Theory of Interferometers — II.

by Ronald W. P. Drever, 13 April 1994

Drever lectured from the attached transparencies. His lecture focussed on optical configurations for interferometers that are more complex than the simple interferometers of Thorne's lecture:

- A *recycling interferometer* (more normally called *power recycling interferometer* these days), in which the light going back from the interferometer toward the laser gets recycled back into the interferometer together with and in phase with new laser light. Such power recycling (invented in the early 1980s by Drever) will be used in LIGO's first interferometers.
- A *dual recycling interferometer* (also called *signal recycling*) in which light power is recycled at the laser's port of the interferometer, and the gravity-wave signal is recycled (and thereby enchanced) at the photodetector's port. Such signal recycling can be used to enhance the interferometer's performance in the vicinity of (most) any desired gravity-wave frequency and over (most) any desired bandwidth around that frequency. It is likely to find application, for example, in deep searches for the gravitational waves from pulsars. A dual recycled interferometers has the added benefit of less sensitivity to irregularities in the mirrors and beam splitter than an ordinary interferometer.
- A *resonant recycling interferometer*, a configuration that accomplishes the same thing as signal recycling but in a more complicated and less practical way. (This configuration was invented by Drever in the early 1980s; signal recycling is an improvement on it, devised by Brian Meers in the late 1980s.)
- A *doubly resonant signal recycling interferometer*. This configuration, invented by Drever and Meers, recycles (and enhances) both signal sidebands of the light's carrier frequency; ordinary signal recycling recycles and enhances only one signal sideband.
- A *resonant sideband extraction interferometer*, whose configuration looks just like that of a dual recycling interferometer but performs quite differently because of a different fine tuning of the location and reflectivity of the recycling mirrors. This configuration, invented recently by M.J. Mizuno (a Japanese graduate student working with the Garching, Germany group) stores the carrier-frequency light in the arms for a long time, while resonantly extracting the signal sideband from the arms after about a half cycle of the gravitational wave. It thereby achieves a broad-band sensitivity comparable to that of a power recycled interferometer, but with far less light power passing through the beam splitter and hence with less problem from high-power heating of the splitter.

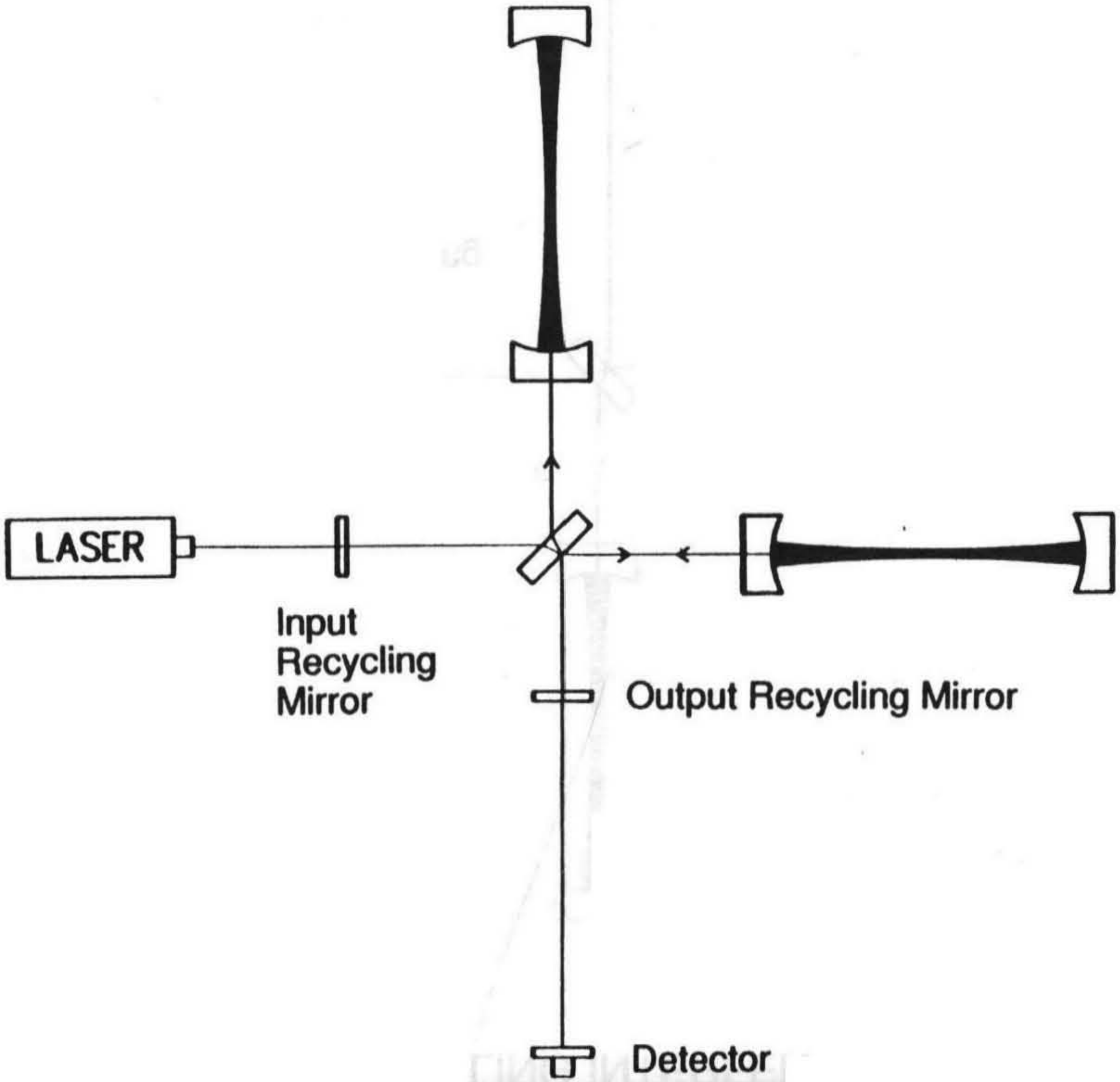
RECYCLING INTERFEROMETER

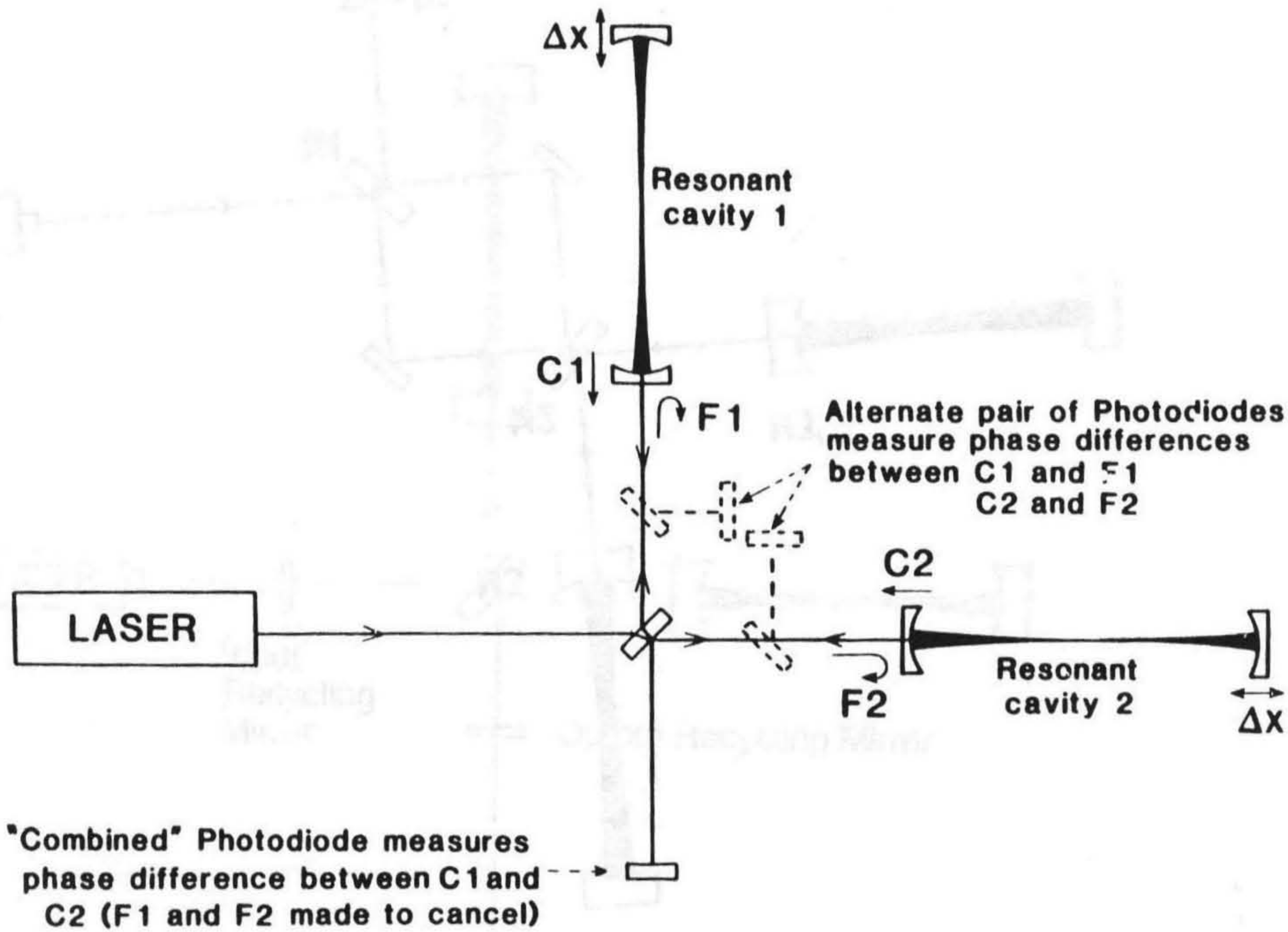


2.

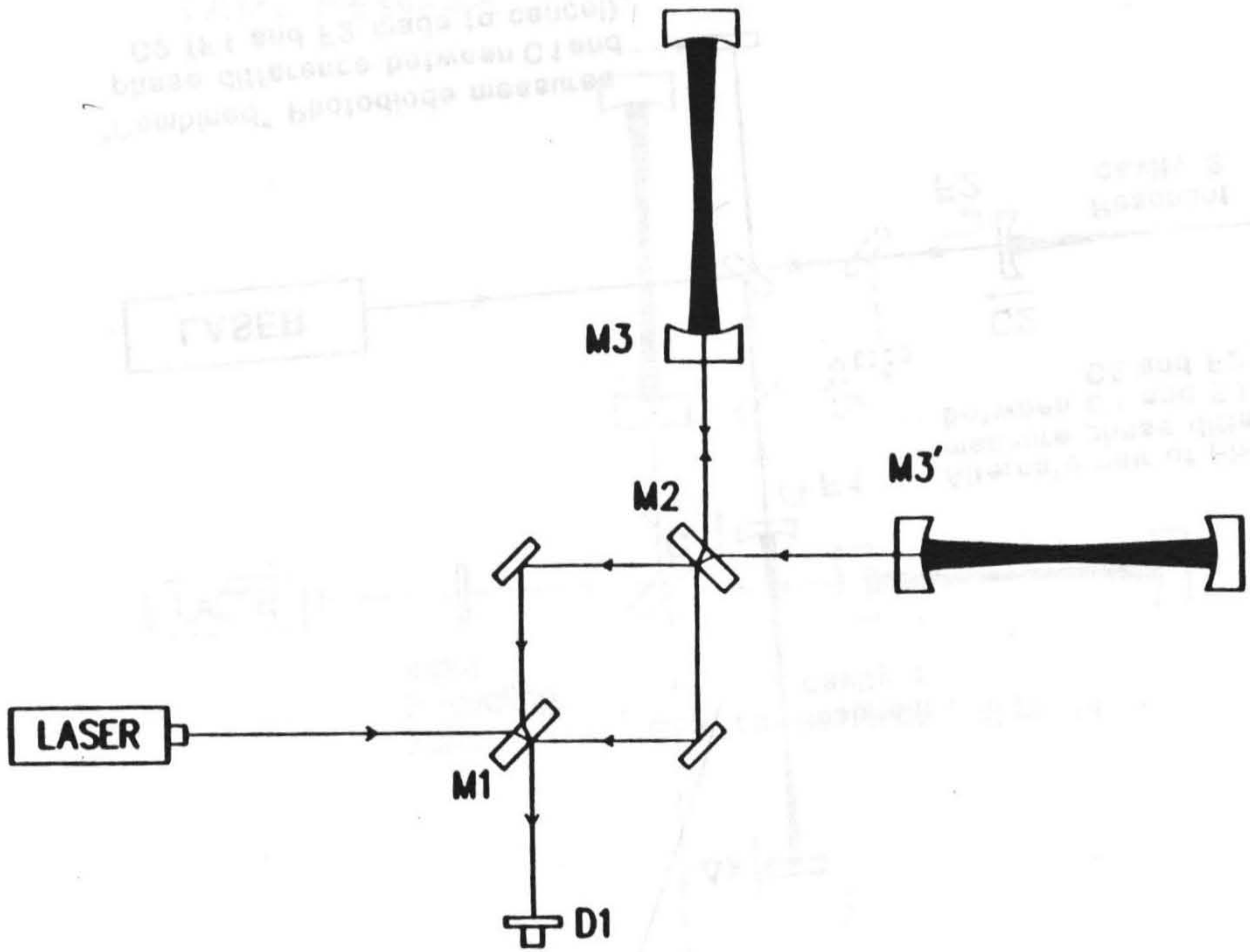
DUAL RECYCLING INTERFEROMETER

(Also Resonant Side Band Extraction Interferometer)

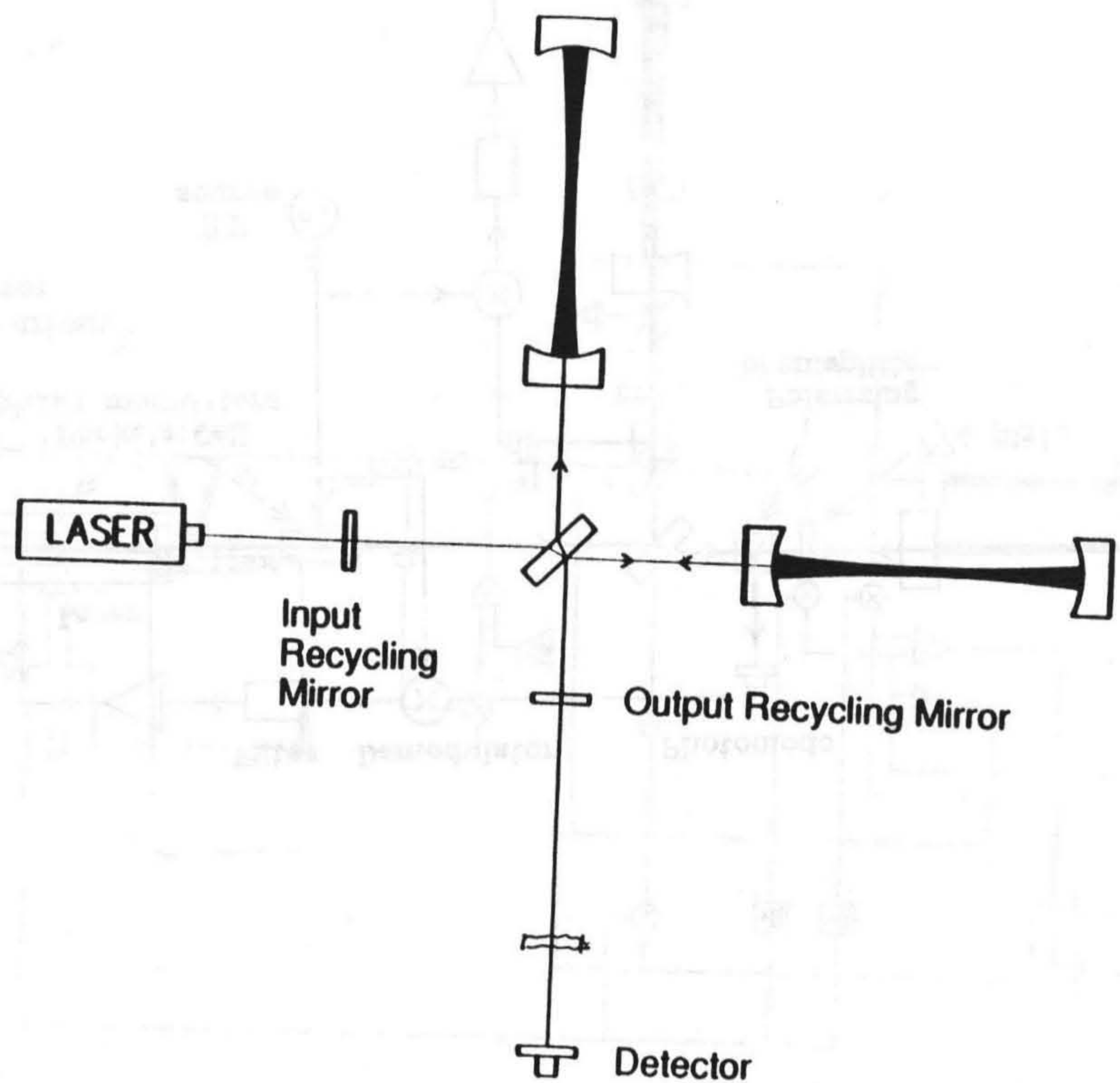


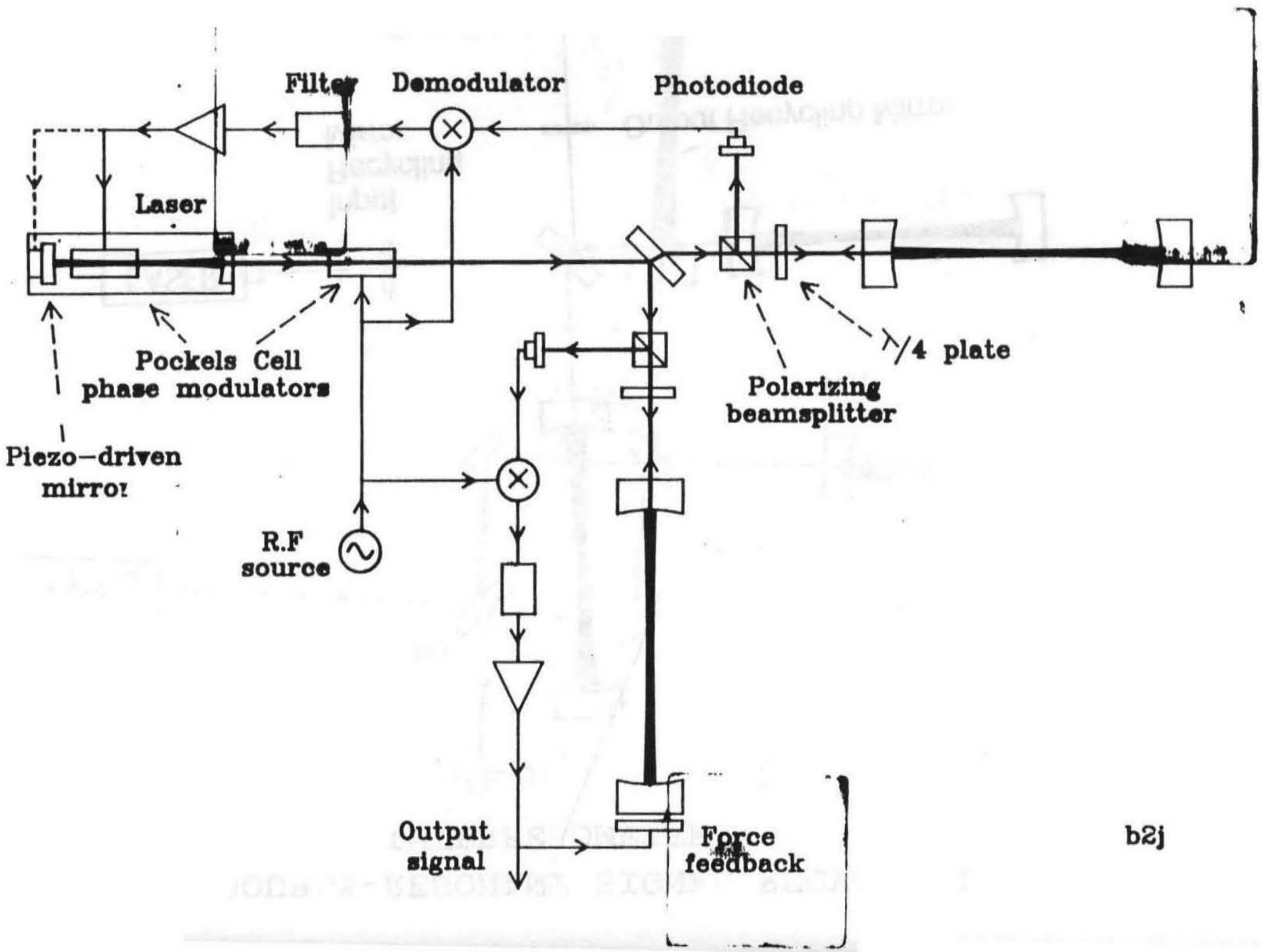

Δx
Resonant cavity 1
C1
F1
Alternate pair of Photodiodes measure phase differences between C1 and F1
C2 and F2
C2
LASER
F2
Resonant cavity 2
Δx
"Combined" Photodiode measures phase difference between C1 and C2 (F1 and F2 made to cancel)

RESONANT RECYCLING INTERFEROMETER

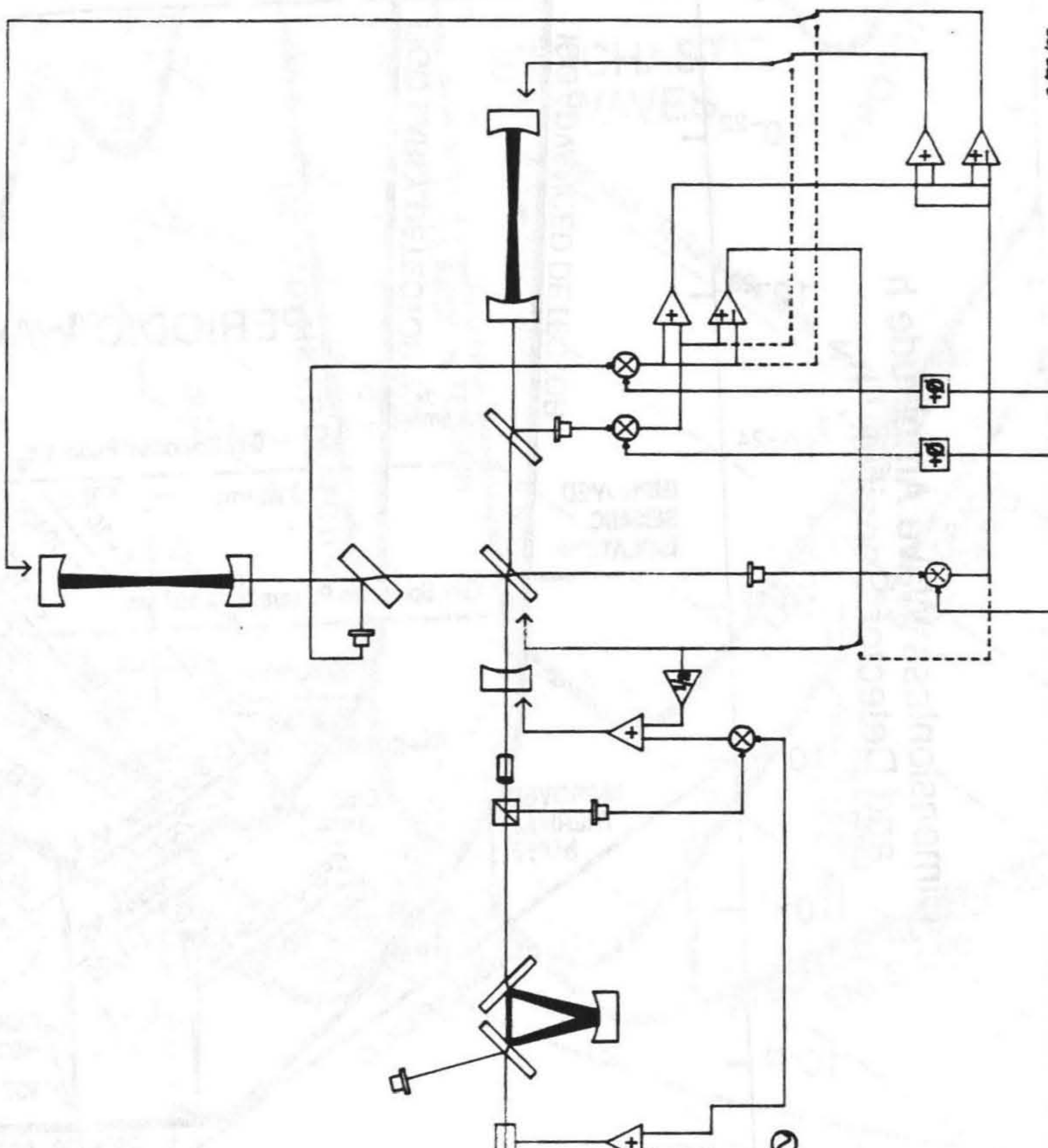


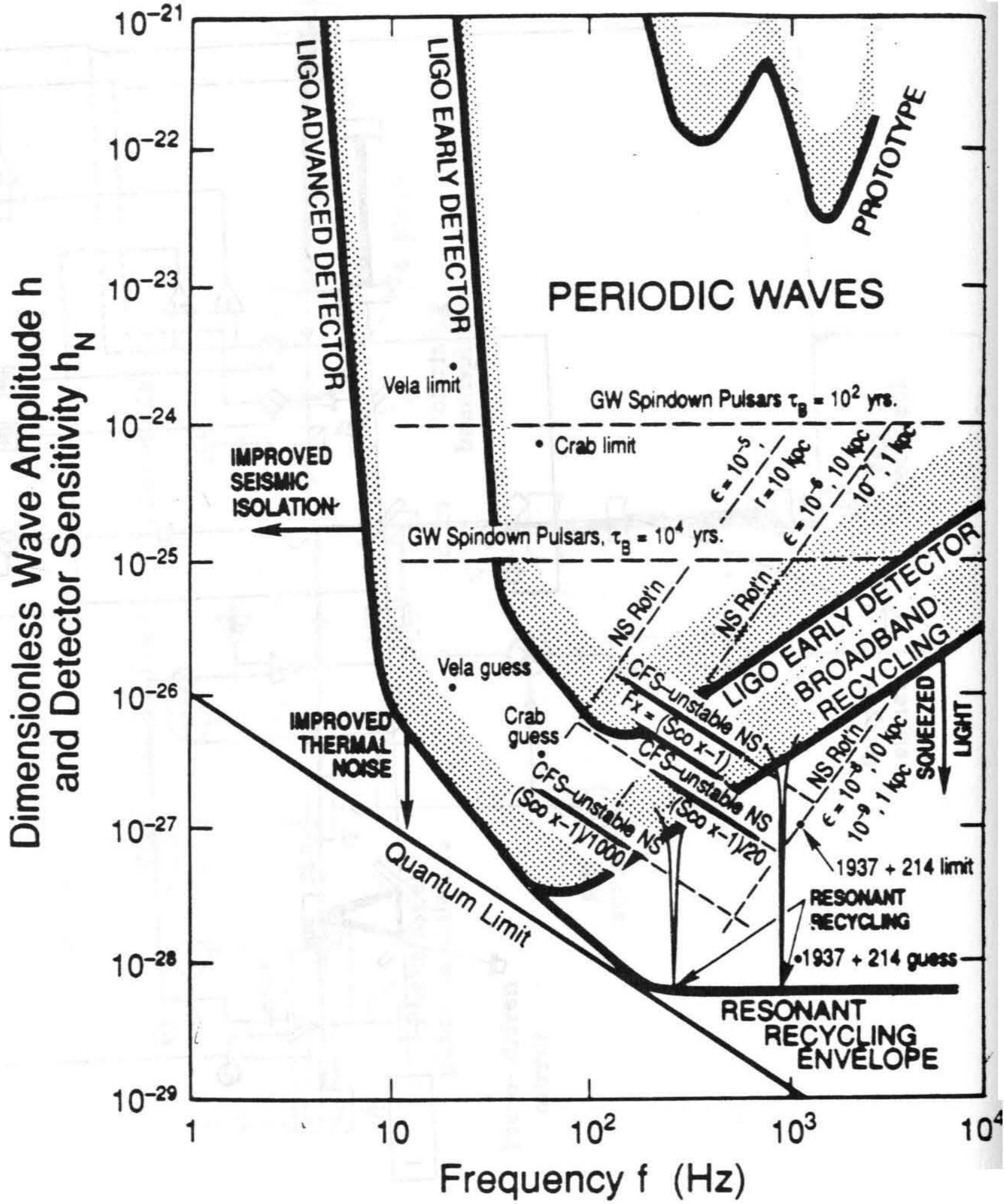
DOUBLY-RESONANT SIGNAL RECYCLING INTERFEROMETER



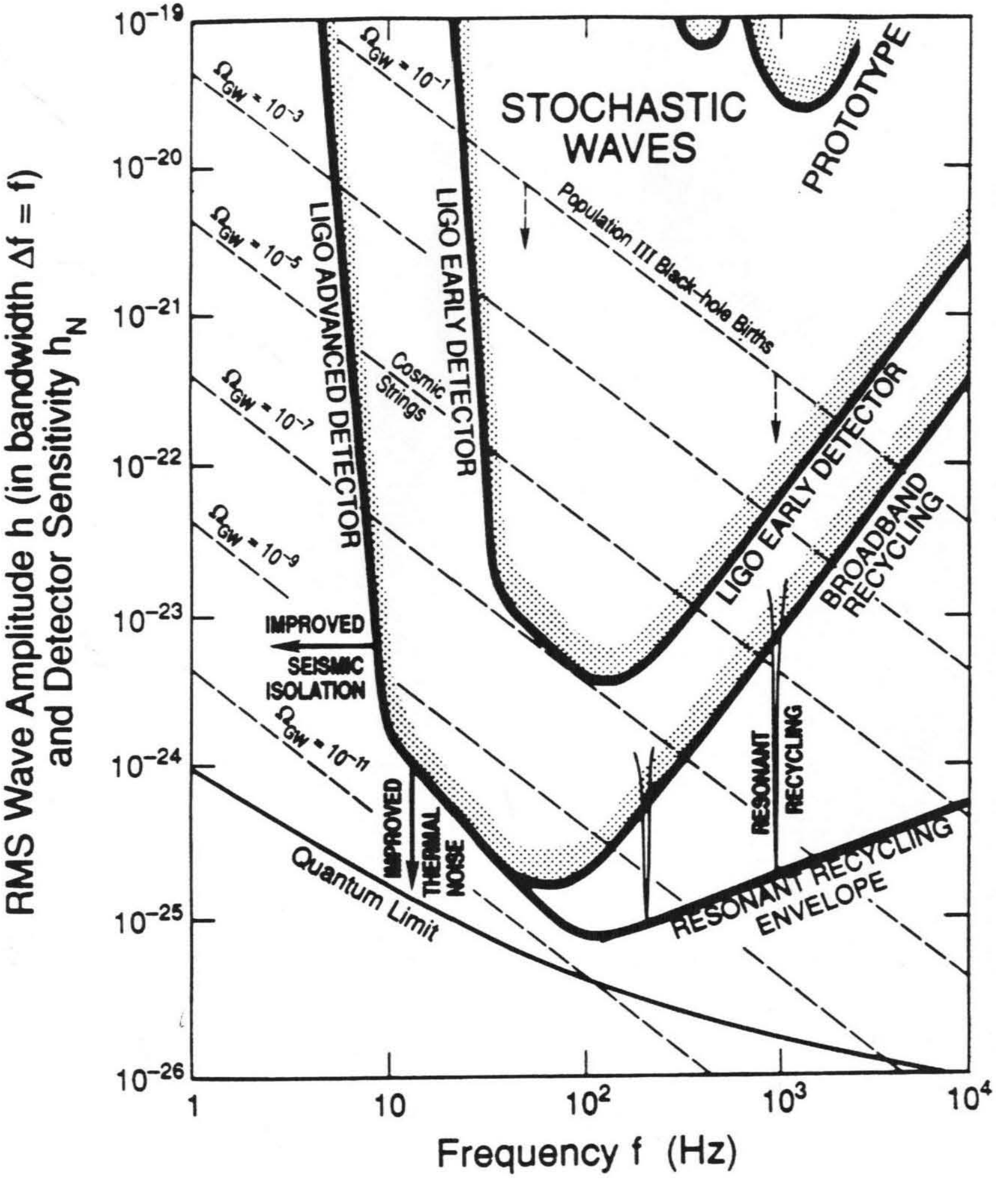

Filter
Demodulator
Photodiode
Laser
Pockels Cell
phase modulators
Piezo-driven
mirror
R.F
source
Polarizing
beamsplitter
λ/4 plate
Output
signal
Force
feedback

b2j





LIGO ADVANCED DETECTOR
LIGO EARLY DETECTOR
PROTOTYPE
PERIODIC WAVES
Vela limit
GW Spindown Pulsars $\tau_B = 10^2$ yrs.
Crab limit
IMPROVED SEISMIC ISOLATION
$\epsilon = 10^{-5}$, r = 10 kpc
$\epsilon = 10^{-6}$, 10 kpc
10^{-7}, 1 kpc
GW Spindown Pulsars, $\tau_B = 10^4$ yrs.
NS Rot'n
NS Rot'n
LIGO EARLY DETECTOR
BROADBAND RECYCLING
Vela guess
CFS–unstable NS: Fx = (Sco x–1)
CFS–unstable NS (Sco x–1)/20
CFS–unstable NS (Sco x–1)/1000
Crab guess
IMPROVED THERMAL NOISE
NS Rot'n
$\epsilon = 10^{-8}$, 10 kpc
10^{-9}, 1 kpc
SQUEEZED LIGHT
1937 + 214 limit
Quantum Limit
RESONANT RECYCLING
•1937 + 214 guess
RESONANT RECYCLING ENVELOPE
Dimensionless Wave Amplitude h and Detector Sensitivity h_N
Frequency f (Hz)
10^{-21}
10^{-22}
10^{-23}
10^{-24}
10^{-25}
10^{-26}
10^{-27}
10^{-28}
10^{-29}
1
10
10^2
10^3
10^4



RMS Wave Amplitude h (in bandwidth Δf = f) and Detector Sensitivity h_N
Frequency f (Hz)
STOCHASTIC WAVES
PROTOTYPE
LIGO EARLY DETECTOR
LIGO ADVANCED DETECTOR
LIGO EARLY DETECTOR
BROADBAND RECYCLING
RESONANT RECYCLING
RESONANT RECYCLING ENVELOPE
Quantum Limit
IMPROVED SEISMIC ISOLATION
IMPROVED THERMAL NOISE
Population III Black-hole Births
Cosmic Strings
Ω_GW = 10^-1
Ω_GW = 10^-3
Ω_GW = 10^-5
Ω_GW = 10^-7
Ω_GW = 10^-9
Ω_GW = 10^-11
10^-19
10^-20
10^-21
10^-22
10^-23
10^-24
10^-25
10^-26
1
10
10^2
10^3
10^4